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Trilocal Structures. III. Expansion in Tree Functions

Roger E. Clapp, Evelyn W. Mack, Fay Simons, and Joseph A. Wolf, Jr.

Basic Research Associates, Incorporated, Cambridge, Massachusetts 02138

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An alternative set of expansion functions is described. The first 48 in the set are listed explicitly, and 16 generalized formulas are given, from which the entire rest-system set of functions can be constructed. Explicit and generalized matrix elements for the rest Hamiltonian are given. An auxiliary condition is introduced, leading to explicit formulas and generalized formulas for the expansion coefficients, C_j, accompanying the expansion functions, φ_j.

1. INTRODUCTION

In the second article in this set (Clapp et al., 1979), which will be referred to as II,[1] a system of trilocal expansion functions was described. Each function in that system contained a "radial" factor of the form $h_n(\kappa\mathfrak{R})$, where the hyperspherical radial variable $\mathfrak{R}$ was defined by

$$\mathfrak{R}^2 = (\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_2 - \mathbf{r}_3)^2 + (\mathbf{r}_3 - \mathbf{r}_1)^2 \tag{1.1}$$

Because of the way this variable encloses the three points, the functions will be characterized as "bowl" functions.

Most "bowl" functions also depend explicitly on the relative vectors $\mathbf{r}$ and $\boldsymbol{\rho}$. These were defined earlier, in the first article in this set (Clapp et al., 1980), to be referred to as I. The definitions are

$$\mathbf{r} = (2\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_3)/2, \qquad \boldsymbol{\rho} = (\mathbf{r}_2 - \mathbf{r}_3) \tag{1.2}$$

[1]Equations from II will be referred to as (II.1.1) etc.

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Insertion of (1.2) into (1.1) gives

$$\mathcal{R}^2 = 2r^2 + 3\rho^2/2 \tag{1.3}$$

The present article will introduce an alternative expansion, in terms of "tree" functions. Here the "radial" part of each function is contained in a factor of the form $j_{m,n}$. This factor is itself the product of two factors, with $j_{0,0}$ being given by

$$j_{0,0} = j_0(\kappa_r r) j_0(\kappa_\rho \rho) \tag{1.4}$$

where each j_0 on the right is an ordinary spherical Bessel function of order zero. In (1.4), and in the more general $j_{m,n}$ to be defined later, the "radial" dependence is separated into a factor depending on the magnitude of $\mathbf{r}$ and a second factor depending on the magnitude of $\boldsymbol{\rho}$. The branching relationship of these two vectors is evident from Figure 1, which also shows the centroid vector $\mathbf{R}$, defined by

$$\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)/3 \tag{1.5}$$

The "tree" designation for a function containing a factor $j_{m,n}$ is meant to suggest this branching.

The two expansions ought to be equivalent, so that it should just be a matter of convenience which one is chosen. The "bowl" functions display the symmetry properties in a more self-evident way. On the other hand, the "tree" functions can be easier to work with, and the resulting structures can lend themselves to more straightforward visualizations.

For the next few articles, we will put aside the "bowl" functions and work with the "tree" functions. Later we can return, and look for the

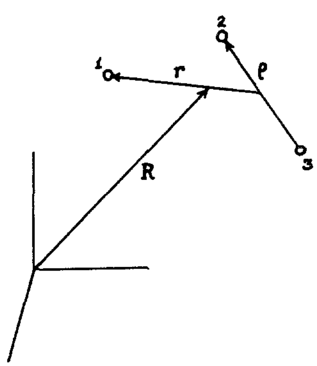

Fig. 1. Coordinate vectors for the three-body problem.

equivalence. It is likely that both kinds of expansion functions will be needed for the efficient treatment of more complicated N-local systems, particularly those in which a closed shell is combined with a few additional quanta. The "bowl" functions would appear to be most appropriate for the description of the closed shell, while the "tree" functions are used to connect the shell to the added individual quanta.

2. REST SYSTEM EXPANSION

As in the preceding article, we will direct our attention to the upper middle tier, characterized by $M_\tau = +1/2$. The initial function in the expansion, replacing $\psi_1^{+1/2}$ in (II.2.2), will here be defined by

$$\varphi_1 = N_0 j_{0,0}\left[(+)^{\tau\,2b}(1) + (-)^{\tau\,2c}(1)\right] \tag{2.1}$$

where the σ-spin functions ${}^{2b}(1)$ and ${}^{2c}(1)$ are those given explicitly in (II.2.9) and (II.2.10). The τ-spin functions to be used here are

$$(+)^\tau = \begin{array}{|c|}\hline 0\\ \hline 0\\ 1\\ 1\\ \hline 0\\ 0\\ 0\\ \hline 0\\ \hline\end{array}^{\,\tau} \qquad (-)^\tau = \begin{array}{|c|}\hline 0\\ \hline 0\\ 1\\ -1\\ \hline 0\\ 0\\ 0\\ \hline 0\\ \hline\end{array}^{\,\tau} \tag{2.2}$$

in the notational format used in the preceding article. The normalization factor in (2.1) is given by

$$N_0 = \kappa^{3/2}/4 \tag{2.3}$$

and differs by a numerical factor from the N_0 given previously in (II.2.3), but has the same role of giving each function in the expansion the dimension of $(\text{length})^{-3/2}$ so that its square can be interpreted as a volume density.

The action of the trilocal Hamiltonian upon φ_1 will generate the other functions in the tree expansion. Each will have a "radial" factor $j_{m,n}$ which can be represented in terms of the spherical Bessel functions through the definition

$$j_{m,n} = (\kappa_r r)^{-m} j_m(\kappa_r r)(\kappa_\rho \rho)^{-n} j_n(\kappa_\rho \rho) \tag{2.4}$$

The factors (2.4) satisfy the two recursion relations

$$j_{m,n} = (2m+1)^{-1}\left[j_{m-1,n} + \kappa_r^2 r^2 j_{m+1,n} \right] \tag{2.5a}$$

$$j_{m,n} = (2n+1)^{-1}\left[j_{m,n-1} + \kappa_\rho^2 \rho^2 j_{m,n+1} \right] \tag{2.5b}$$

and the two differentiation relations

$$\nabla_r j_{m,n} = -\kappa_r^2 \mathbf{r} j_{m+1,n} \tag{2.6a}$$

$$\nabla_\rho j_{m,n} = -\kappa_\rho^2 \boldsymbol{\rho} j_{m,n+1} \tag{2.6b}$$

The τ-spin functions (2.2) make reference to a basis system of the form (II.2.8), but with the superscript σ replaced by a superscript τ. It can be seen that only two τ-spin combinations are utilized, and for each of these we have

$$\tau_{1\zeta} = +1 \tag{2.7}$$

We can also see that the operators $\tau_{2\zeta}$ and $\tau_{3\zeta}$ interchange the functions (2.2), according to the pattern given by

$$\tau_{2\zeta}(+)^\tau = -(-)^\tau, \qquad \tau_{2\zeta}(-)^\tau = -(+)^\tau \tag{2.8}$$

$$\tau_{3\zeta}(+)^\tau = (-)^\tau, \qquad \tau_{3\zeta}(-)^\tau = (+)^\tau \tag{2.9}$$

We can define an exchange operator P^τ through

$$P^\tau(+)^\tau = (-)^\tau, \qquad P^\tau(-)^\tau = (+)^\tau \tag{2.10}$$

This operator is similar in its effect to the operator P^τ defined in equation (13.4) of Clapp (1980), for the bilocal analysis, and we anticipate that the pair of quanta involved in the operations in (2.10) above may play the role of an attached bilocal photon within the trilocal structure.

The Hamiltonian operator, specialized to the upper-middle tier and to the τ-spin dependence in (2.2), can be written as

$$H = H_k + H_{r,\rho} \tag{2.11a}$$

$$H_k = (1/9)(\boldsymbol{\sigma}^s \cdot \mathbf{k} - 3P^\tau \boldsymbol{\sigma}^b \cdot \mathbf{k} + \boldsymbol{\sigma}^c \cdot \mathbf{k}) \tag{2.11b}$$

$$H_{r,\rho} = (1/6i\kappa)\left(2\boldsymbol{\sigma}^s \cdot \nabla_r - 4P^\tau \boldsymbol{\sigma}^s \cdot \nabla_\rho + 3P^\tau \boldsymbol{\sigma}^b \cdot \nabla_r + 2\boldsymbol{\sigma}^c \cdot \nabla_r + 2P^\tau \boldsymbol{\sigma}^c \cdot \nabla_\rho\right) \tag{2.11c}$$

These relations are equivalent to (II.2.22), and are based on (I.2.8a). The spin operator $\boldsymbol{\sigma}^s$ is the one which was defined in (II.2.21), while $\boldsymbol{\sigma}^b$ and $\boldsymbol{\sigma}^c$ are defined by

$$\boldsymbol{\sigma}^b = \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3 \tag{2.12a}$$

$$\boldsymbol{\sigma}^c = 2\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3 \tag{2.12b}$$

Operation upon φ_1 by the rest-system Hamiltonian $H_{r,\rho}$ will generate a set of functions which do not involve the momentum $\mathbf{k}$, together with matrix elements linking these functions. The procedure is similar to that used in II for generating the bowl functions ψ_j from the initial function ψ_1.

As in II, the relative normalization of the individual functions is determined by the requirement that the matrix form of the operator (2.11c) be symmetrical about its main diagonal. Appendix A contains a listing of the first 48 functions φ_j in the tree expansion. Appendix B contains the first 24 rows of the matrix representation of the operator $H_{r,\rho}$ defined in (2.11c).

The functions φ_j utilize the same 16 σ-spin rotational functions that were introduced in Clapp (1961) and were listed in (II.4.1). It can furthermore be seen from Appendix A that the functions follow a pattern having a cycle length of 16. These functions can in fact be given in the generalized form shown in Appendix C, where the P_N are Legendre polynomials.

In the generalized formulation, the index N is used to count the number of cycles, each cycle containing 16 functions. Setting $N=0$ gives the functions φ_1 through φ_{16}, as can be seen from a comparison of those functions as given explicitly in Appendix A with the formulas in Appendix C.

The mechanics of operating with the terms in (2.11c) upon the functions in Appendix A are straightforward but somewhat tedious. Considerable use is made of the relationships assembled in Appendix D of Clapp (1961). The algebraic operations can also be carried out directly upon the generalized functions in Appendix C, and this leads to the generalized matrix elements included here in Appendix D. The consistency between the generalized and explicit rows of the $H_{r,\rho}$ matrix can be verified through substitution of particular values of N into the equations of Appendix D.

3. AN AUXILIARY CONDITION

In the bilocal analysis (Clapp, 1980), the use of an auxiliary operator Q_λ was important in the reduction of infinite equation systems to finite systems. Auxiliary operators will also be important in the trilocal analysis. There are several operators which commute with the Hamiltonian and can therefore be expected to be conserved.

One candidate for conservation is a quadratic differential operator obtained from the two relative gradients and the exchange operator P^τ. An auxiliary condition will be introduced through the operator equation

$$P^\tau(\nabla_r \cdot \nabla_\rho)\Phi = -\kappa_r \kappa_\rho \nu \Phi \tag{3.1}$$

which then serves to define the eigenvalue ν. The wave function Φ will here be assumed to have the expanded form

$$\Phi = C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3 + \cdots \tag{3.2}$$

While the complete wave function will contain momentum-dependent functions, not included in the rest-system functions of Appendices A and C, these will not be considered for the present. The operator in (3.1) does not couple between rest-system functions and momentum-dependent functions.

This operator, in fact, couples only within limited families of functions, as a consequence of its scalar form which contains no σ-spin operators. For example, the action of this operator upon the function φ_1 gives

$$P^\tau(\nabla_r \cdot \nabla_\rho)\varphi_1 = (1/3)^{1/2}\kappa_r\kappa_\rho\varphi_{17} \tag{3.3}$$

and action upon φ_{17} gives

$$P^\tau(\nabla_r \cdot \nabla_\rho)\varphi_{17} = \kappa_r\kappa_\rho\left[(1/3)^{1/2}\varphi_1 + (4/15)^{1/2}\varphi_{33}\right] \tag{3.4}$$

Substitution of (3.2) into (3.1), followed by rearrangement of terms, leads to relationships among the C_j which include

$$-\nu C_1 = (1/3)^{1/2} C_{17} \tag{3.5a}$$

$$-\nu C_{17} = (1/3)^{1/2} C_1 + (4/15)^{1/2} C_{33} \tag{3.5b}$$

and so forth. These can be solved to give

$$C_{17} = -3^{1/2} \nu C_1 \tag{3.6a}$$

$$C_{33} = 5^{1/2}(1/2)(3\nu^2 - 1) C_1 \tag{3.6b}$$

and the sequence can obviously be continued indefinitely so that all of the coefficients of the form C_{16N+1} can be expressed as multiples of C_1.

From the first few expressions, it is evident that there is a general form, using the Legendre polynomials P_N,

$$C_{16N+1} = (2N+1)^{1/2} P_N(-\nu) C_1 \tag{3.7}$$

and this form can be verified through operation by $P^{\tau}(\nabla_r \cdot \nabla_\rho)$ upon φ_{16N+1} as given in Appendix C. A similar formula gives C_{16N+2} in terms of C_2.

The same procedure can be used with all of the other expansion functions. When this is done, however, certain subsets are found to be grouped together into families that give formulas that are somewhat more complicated than (3.6) and (3.7). The results of this procedure have been assembled into Appendices E and F. Appendix E contains explicit formulas giving the coefficients C_{17} through C_{48} in terms of C_1 through C_{16}, while Appendix F contains the corresponding generalized formulas giving all of the higher coefficients in terms of the first 16.

Comparisons between the C_j formulas (Appendices E and F) and the φ_j formulas (Appendices A and C) show a strict parallelism. Once the key has been spelled out, it is possible to move back and forth between a C_j formula and the corresponding φ_j formula. A similar parallelism was found in the bilocal analysis (Clapp, 1980), where it led to the translation from expanded wave functions into closed-form solutions.

4. SUMMARY

A system of expansion functions has been constructed, similar in some ways to the functions described in II. The primary difference is in the "radial" dependence. In the bowl functions of II, this was contained in a hyperspherical Bessel function of a single radial variable. In the tree functions described here, it is contained in the product of two spherical Bessel functions, of two radial variables related through the branched construction in Figure 1.

The tree functions arrange themselves into cycles of 16, and generalized formulas for functions, for matrix elements, and for expansion coefficients, incorporating this cycle structure, have been assembled into Appendices C, D, and F. The formulas for the expansion coefficients C_j arise out of the imposition of an auxiliary condition, equation (3.1), which introduces a new parameter ν, a dimensionless parameter whose magnitude will be specified at a later stage of the analysis, after the expansion has been enlarged to include the dependence of the wave function upon the centroid momentum.

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APPENDIX A: 48 TRILOCAL BASIS FUNCTIONS

$$\varphi_1 = N_0 j_{0,0}\left[(+)^{\tau\, 2b}(1) + (-)^{\tau\, 2c}(1)\right]$$

$$\varphi_2 = N_0(1/3)^{1/2} j_{0,0}\left[3(+)^{\tau\, 2b}(1) - (-)^{\tau\, 2c}(1)\right]$$

$$\varphi_3 = iN_0(2/3)^{1/2}\kappa_r j_{1,0}(-)^{\tau\, 4}(\mathbf{r})$$

$$\varphi_4 = iN_0(2/3)^{1/2}\kappa_\rho j_{0,1}(+)^{\tau\, 4}(\boldsymbol{\rho})$$

$$\varphi_5 = iN_0\kappa_r j_{1,0}\left[(+)^{\tau\, 2b}(\mathbf{r}) + (-)^{\tau\, 2c}(\mathbf{r})\right]$$

$$\varphi_6 = iN_0(1/3)^{1/2}\kappa_r j_{1,0}\left[3(+)^{\tau\, 2b}(\mathbf{r}) - (-)^{\tau\, 2c}(\mathbf{r})\right]$$

$$\varphi_7 = iN_0\kappa_\rho j_{0,1}\left[(-)^{\tau\, 2b}(\boldsymbol{\rho}) + (+)^{\tau\, 2c}(\boldsymbol{\rho})\right]$$

$$\varphi_8 = iN_0(1/3)^{1/2}\kappa_\rho j_{0,1}\left[3(-)^{\tau\, 2b}(\boldsymbol{\rho}) - (+)^{\tau\, 2c}(\boldsymbol{\rho})\right]$$

$$\varphi_9 = N_0\kappa_r\kappa_\rho j_{1,1}(+)^{\tau\, 4}(i\mathbf{r}\times\boldsymbol{\rho})$$

$$\varphi_{10} = N_0(6)^{1/2}\kappa_r^2 j_{2,0}(-)^{\tau\, 4}(\mathbf{rr})$$

$$\varphi_{11}=N_0(6)^{1/2}\kappa_\rho^2 j_{0,2}(-)^{\tau\,4}(\boldsymbol{\rho\rho})$$

$$\varphi_{12}=N_0(9/5)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^{\tau\,4}(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})$$

$$\varphi_{13}=N_0(3/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}\left[(-)^{\tau\,2b}(i\mathbf{r}\times\boldsymbol{\rho})+(+)^{\tau\,2c}(i\mathbf{r}\times\boldsymbol{\rho})\right]$$

$$\varphi_{14}=N_0(1/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}\left[3(-)^{\tau\,2b}(i\mathbf{r}\times\boldsymbol{\rho})-(+)^{\tau\,2c}(i\mathbf{r}\times\boldsymbol{\rho})\right]$$

$$\varphi_{15}=iN_0(12)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^{\tau\,4}(i\mathbf{r}\mathbf{r}\times\boldsymbol{\rho})$$

$$\varphi_{16}=iN_0(12)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^{\tau\,4}(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})$$

$$\varphi_{17}=N_0(3)^{1/2}\kappa_r\kappa_\rho j_{1,1}(\mathbf{r}\cdot\boldsymbol{\rho})\left[(-)^{\tau\,2b}(1)+(+)^{\tau\,2c}(1)\right]$$

$$\varphi_{18}=N_0\kappa_r\kappa_\rho j_{1,1}(\mathbf{r}\cdot\boldsymbol{\rho})\left[3(-)^{\tau\,2b}(1)-(+)^{\tau\,2c}(1)\right]$$

$$\varphi_{19}=iN_0(3)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^{\tau}\left[{}^4(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(\boldsymbol{\rho})r^2/3\right]$$

$$\varphi_{20}=iN_0(3)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^{\tau}\left[{}^4(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(\mathbf{r})\rho^2/3\right]$$

$$\varphi_{21}=iN_0(9/2)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}$$
$$\cdot\left\{(-)^{\tau}\left[{}^{2b}(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^{2b}(\boldsymbol{\rho})r^2/3\right]+(+)^{\tau}\left[{}^{2c}(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^{2c}(\boldsymbol{\rho})r^2/3\right]\right\}$$

$$\varphi_{22}=iN_0(3/2)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}$$
$$\cdot\left\{3(-)^{\tau}\left[{}^{2b}(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^{2b}(\boldsymbol{\rho})r^2/3\right]-(+)^{\tau}\left[{}^{2c}(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^{2c}(\boldsymbol{\rho})r^2/3\right]\right\}$$

$$\varphi_{23}=iN_0(9/2)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}$$
$$\cdot\left\{(+)^{\tau}\left[{}^{2b}(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^{2b}(\mathbf{r})\rho^2/3\right]+(-)^{\tau}\left[{}^{2c}(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^{2c}(\mathbf{r})\rho^2/3\right]\right\}$$

$$\varphi_{24}=iN_0(3/2)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}$$
$$\cdot\left\{3(+)^{\tau}\left[{}^{2b}(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^{2b}(\mathbf{r})\rho^2/3\right]-(-)^{\tau}\left[{}^{2c}(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^{2c}(\mathbf{r})\rho^2/3\right]\right\}$$

$$\varphi_{25}=N_0(5)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^{\tau\,4}(i\mathbf{r}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})$$

$$\varphi_{26}=N_0(30)^{1/2}\kappa_r^3\kappa_\rho j_{3,1}(+)^{\tau}\left[{}^4(\mathbf{r}\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})r^2/5\right]$$

$$\varphi_{27}=N_0(30)^{1/2}\kappa_r\kappa_\rho^3 j_{1,3}(+)^{\tau}\left[{}^4(\boldsymbol{\rho\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})\rho^2/5\right]$$

$$\varphi_{28}=N_0(135/7)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau$$
$$\times\left[{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(\mathbf{rr})2\rho^2/3-{}^4(\boldsymbol{\rho\rho})2r^2/3\right]$$

$$\varphi_{29}=N_0(15/2)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(\mathbf{r}\cdot\boldsymbol{\rho})\left[(+)^{\tau\,2b}(i\mathbf{r}\times\boldsymbol{\rho})+(-)^{\tau\,2c}(i\mathbf{r}\times\boldsymbol{\rho})\right]$$

$$\varphi_{30}=N_0(5/2)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(\mathbf{r}\cdot\boldsymbol{\rho})\left[3(+)^{\tau\,2b}(i\mathbf{r}\times\boldsymbol{\rho})-(-)^{\tau\,2c}(i\mathbf{r}\times\boldsymbol{\rho})\right]$$

$$\varphi_{31}=iN_0(75)^{1/2}\kappa_r^3\kappa_\rho^2 j_{3,2}(-)^\tau\left[{}^4(i\mathbf{rr}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})r^2/5\right]$$

$$\varphi_{32}=iN_0(75)^{1/2}\kappa_r^2\kappa_\rho^3 j_{2,3}(+)^\tau\left[{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{rr}\times\boldsymbol{\rho})\rho^2/5\right]$$

$$\varphi_{33}=N_0(45/4)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/3\right]\cdot\left[(+)^{\tau\,2b}(1)+(-)^{\tau\,2c}(1)\right]$$

$$\varphi_{34}=N_0(15/4)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/3\right]\cdot\left[3(+)^{\tau\,2b}(1)-(-)^{\tau\,2c}(1)\right]$$

$$\varphi_{35}=iN_0(25/2)^{1/2}\kappa_r^3\kappa_\rho^2 j_{3,2}(-)^\tau\cdot\left\{{}^4(\mathbf{r})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^4(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})2r^2/5\right\}$$

$$\varphi_{36}=iN_0(25/2)^{1/2}\kappa_r^2\kappa_\rho^3 j_{2,3}(+)^\tau\cdot\left\{{}^4(\boldsymbol{\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^4(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})2\rho^2/5\right\}$$

$$\varphi_{37}=iN_0(75/4)^{1/2}\kappa_r^3\kappa_\rho^2 j_{3,2}\cdot\Big((+)^\tau\Big\{{}^{2b}(\mathbf{r})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]$$
$$-{}^{2b}(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})2r^2/5\Big\}+(-)^\tau\Big\{{}^{2c}(\mathbf{r})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^{2c}(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})2r^2/5\Big\}\Big)$$

$$\varphi_{38}=iN_0(5/2)\kappa_r^3\kappa_\rho^2 j_{3,2}\cdot\Big(3(+)^\tau\Big\{{}^{2b}(\mathbf{r})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^{2b}(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})2r^2/5\Big\}$$
$$-(-)^\tau\Big\{{}^{2c}(\mathbf{r})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^{2c}(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})2r^2/5\Big\}\Big)$$

$$\varphi_{39}=iN_0(75/4)^{1/2}\kappa_r^2\kappa_\rho^3 j_{2,3}\cdot\Big((-)^\tau\Big\{{}^{2b}(\boldsymbol{\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]$$
$$-{}^{2b}(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})2\rho^2/5\Big\}+(+)^\tau\Big\{{}^{2c}(\boldsymbol{\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^{2c}(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})2\rho^2/5\Big\}\Big)$$

$$\varphi_{40}=iN_0(5/2)\kappa_r^2\kappa_\rho^3 j_{2,3}\cdot\Big(3(-)^\tau\Big\{{}^{2b}(\boldsymbol{\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^{2b}(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})2\rho^2/5\Big\}$$
$$-(+)^\tau\Big\{{}^{2c}(\boldsymbol{\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^{2c}(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})2\rho^2/5\Big\}\Big)$$

$$\varphi_{41}=N_0(175/8)^{1/2}\kappa_r^3\kappa_\rho^3 j_{3,3}(+)^\tau\cdot\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]{}^4(i\mathbf{r}\times\boldsymbol{\rho})$$

$$\varphi_{42} = N_0(525/4)^{1/2}\kappa_r^4\kappa_\rho^2 j_{4,2}(-)^\tau$$
$$\cdot\left\{ {}^4(\mathbf{rr})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - r^2\rho^2/7\right] - {}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})2r^2/7 + {}^4(\boldsymbol{\rho\rho})2r^4/35\right\}$$

$$\varphi_{43} = N_0(525/4)^{1/2}\kappa_r^2\kappa_\rho^4 j_{2,4}(-)^\tau$$
$$\cdot\left\{ {}^4(\boldsymbol{\rho\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - r^2\rho^2/7\right] - {}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})2\rho^2/7 + {}^4(\mathbf{rr})2\rho^4/35\right\}$$

$$\varphi_{44} = N_0(875/8)^{1/2}\kappa_r^3\kappa_\rho^3 j_{3,3}(+)^\tau$$
$$\cdot\left\{ {}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - r^2\rho^2/25\right] - {}^4(\mathbf{rr})(\mathbf{r}\cdot\boldsymbol{\rho})4\rho^2/5 - {}^4(\boldsymbol{\rho\rho})(\mathbf{r}\cdot\boldsymbol{\rho})4r^2/5\right\}$$

$$\varphi_{45} = N_0(525/16)^{1/2}\kappa_r^3\kappa_\rho^3 j_{3,3}\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - r^2\rho^2/5\right]$$
$$\cdot\left[(-)^{\tau\,2b}(i\mathbf{r}\times\boldsymbol{\rho}) + (+)^{\tau\,2c}(i\mathbf{r}\times\boldsymbol{\rho})\right]$$

$$\varphi_{46} = N_0(175/16)^{1/2}\kappa_r^3\kappa_\rho^3 j_{3,3}\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - r^2\rho^2/5\right]$$
$$\cdot\left[3(-)^{\tau\,2b}(i\mathbf{r}\times\boldsymbol{\rho}) - (+)^{\tau\,2c}(i\mathbf{r}\times\boldsymbol{\rho})\right]$$

$$\varphi_{47} = iN_0(735/2)^{1/2}\kappa_r^4\kappa_\rho^3 j_{4,3}(+)^\tau$$
$$\cdot\left\{ {}^4(i\mathbf{rr}\times\boldsymbol{\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - r^2\rho^2/7\right] - {}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})2r^2/7\right\}$$

$$\varphi_{48} = iN_0(735/2)^{1/2}\kappa_r^3\kappa_\rho^4 j_{3,4}(-)^\tau$$
$$\cdot\left\{ {}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - r^2\rho^2/7\right] - {}^4(i\mathbf{rr}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})2\rho^2/7\right\}$$

APPENDIX B: 24 ROWS OF THE $H_{r,\rho}$ MATRIX

$$H_{r,\rho}\varphi_1 = (\kappa_r/\kappa)\left[-(6^{1/2}/2)\varphi_3 + (1/2)\varphi_5 + (3^{1/2}/2)\varphi_6\right]$$
$$+(\kappa_\rho/\kappa)\left[-(6^{1/2}/3)\varphi_4 - \varphi_7 + (3^{1/2}/3)\varphi_8\right]$$
$$H_{r,\rho}\varphi_2 = (\kappa_r/\kappa)\left[-(2^{1/2}/6)\varphi_3 + (3^{1/2}/2)\varphi_5 + (1/6)\varphi_6\right]$$
$$+(\kappa_\rho/\kappa)\left[(2^{1/2}/3)\varphi_4 + (3^{1/2}/3)\varphi_7 - (1/3)\varphi_8\right]$$

$$H_{r,\rho}\varphi_3=(\kappa_r/\kappa)[-(6^{1/2}/2)\varphi_1-(2^{1/2}/6)\varphi_2-(1/3)\varphi_{10}]$$
$$+(\kappa_\rho/\kappa)[-5(6^{1/2}/9)\varphi_9+(30^{1/2}/9)\varphi_{12}+(1/3)\varphi_{13}$$
$$-(3^{1/2}/9)\varphi_{14}+(2^{1/2}/3)\varphi_{17}-(6^{1/2}/9)\varphi_{18}]$$

$$H_{r,\rho}\varphi_4=(\kappa_r/\kappa)[-5(6^{1/2}/18)\varphi_9-(30^{1/2}/18)\varphi_{12}-(1/2)\varphi_{13}$$
$$-(3^{1/2}/18)\varphi_{14}+(2^{1/2}/2)\varphi_{17}+(6^{1/2}/18)\varphi_{18}]$$
$$+(\kappa_\rho/\kappa)[-(6^{1/2}/3)\varphi_1+(2^{1/2}/3)\varphi_2+(2/3)\varphi_{11}]$$

$$H_{r,\rho}\varphi_5=(\kappa_r/\kappa)[(1/2)\varphi_1+(3^{1/2}/2)\varphi_2+(6^{1/2}/2)\varphi_{10}]$$
$$+(\kappa_\rho/\kappa)[(1/3)\varphi_9+(5^{1/2}/3)\varphi_{12}-(6^{1/2}/3)\varphi_{13}$$
$$+(2^{1/2}/3)\varphi_{14}+(3^{1/2}/3)\varphi_{17}-(1/3)\varphi_{18}]$$

$$H_{r,\rho}\varphi_6=(\kappa_r/\kappa)[(3^{1/2}/2)\varphi_1+(1/6)\varphi_2+(2^{1/2}/6)\varphi_{10}]$$
$$+(\kappa_\rho/\kappa)[-(3^{1/2}/9)\varphi_9-(15^{1/2}/9)\varphi_{12}+(2^{1/2}/3)\varphi_{13}$$
$$-(6^{1/2}/9)\varphi_{14}-(1/3)\varphi_{17}+(3^{1/2}/9)\varphi_{18}]$$

$$H_{r,\rho}\varphi_7=(\kappa_r/\kappa)[-(1/2)\varphi_9+(5^{1/2}/2)\varphi_{12}-(6^{1/2}/6)\varphi_{13}$$
$$-(2^{1/2}/2)\varphi_{14}-(3^{1/2}/6)\varphi_{17}-(1/2)\varphi_{18}]$$
$$+(\kappa_\rho/\kappa)[-\varphi_1+(3^{1/2}/3)\varphi_2+(6^{1/2}/3)\varphi_{11}]$$

$$H_{r,\rho}\varphi_8=(\kappa_r/\kappa)[-(3^{1/2}/18)\varphi_9+(15^{1/2}/18)\varphi_{12}-(2^{1/2}/2)\varphi_{13}$$
$$-(6^{1/2}/18)\varphi_{14}-(1/2)\varphi_{17}-(3^{1/2}/18)\varphi_{18}]$$
$$+(\kappa_\rho/\kappa)[(3^{1/2}/3)\varphi_1-(1/3)\varphi_2-(2^{1/2}/3)\varphi_{11}]$$

$$H_{r,\rho}\varphi_9=(\kappa_r/\kappa)[-5(6^{1/2}/18)\varphi_4-(1/2)\varphi_7-(3^{1/2}/18)\varphi_8$$
$$+(3^{1/2}/6)\varphi_{15}-5(3^{1/2}/18)\varphi_{19}-(2^{1/2}/4)\varphi_{21}-(6^{1/2}/36)\varphi_{22}]$$
$$+(\kappa_\rho/\kappa)[-5(6^{1/2}/9)\varphi_3+(1/3)\varphi_5-(3^{1/2}/9)\varphi_6-(3^{1/2}/3)\varphi_{16}$$
$$-5(3^{1/2}/9)\varphi_{20}+(2^{1/2}/6)\varphi_{23}-(6^{1/2}/18)\varphi_{24}]$$

$$H_{r,\rho}\varphi_{10}=(\kappa_r/\kappa)\left[-(1/3)\varphi_3+(6^{1/2}/2)\varphi_5+(2^{1/2}/6)\varphi_6\right]$$
$$+(\kappa_\rho/\kappa)\left[2^{1/2}\varphi_{15}-(2^{1/2}/3)\varphi_{19}-(3^{1/2}/3)\varphi_{21}+(1/3)\varphi_{22}\right]$$

$$H_{r,\rho}\varphi_{11}=(\kappa_r/\kappa)\left[(2^{1/2}/2)\varphi_{16}+(2^{1/2}/6)\varphi_{20}-(3^{1/2}/2)\varphi_{23}-(1/6)\varphi_{24}\right]$$
$$+(\kappa_\rho/\kappa)\left[(2/3)\varphi_4+(6^{1/2}/3)\varphi_7-(2^{1/2}/3)\varphi_8\right]$$

$$H_{r,\rho}\varphi_{12}=(\kappa_r/\kappa)\left[-(30^{1/2}/18)\varphi_4+(5^{1/2}/2)\varphi_7+(15^{1/2}/18)\varphi_8\right.$$
$$\left.+(15^{1/2}/10)\varphi_{15}+(15^{1/2}/90)\varphi_{19}-(10^{1/2}/20)\varphi_{21}-(30^{1/2}/180)\varphi_{22}\right]$$
$$+(\kappa_\rho/\kappa)\left[(30^{1/2}/9)\varphi_3+(5^{1/2}/3)\varphi_5-(15^{1/2}/9)\varphi_6+(15^{1/2}/5)\varphi_{16}\right.$$
$$\left.-(15^{1/2}/45)\varphi_{20}-(10^{1/2}/30)\varphi_{23}+(30^{1/2}/90)\varphi_{24}\right]$$

$$H_{r,\rho}\varphi_{13}=(\kappa_r/\kappa)\left[-(1/2)\varphi_4-(6^{1/2}/6)\varphi_7-(2^{1/2}/2)\varphi_8\right.$$
$$\left.-3(2^{1/2}/4)\varphi_{15}-(2^{1/2}/4)\varphi_{19}-(3^{1/2}/6)\varphi_{21}-(1/2)\varphi_{22}\right]$$
$$+(\kappa_\rho/\kappa)\left[(1/3)\varphi_3-(6^{1/2}/3)\varphi_5+(2^{1/2}/3)\varphi_6-(2^{1/2}/2)\varphi_{16}\right.$$
$$\left.+(2^{1/2}/6)\varphi_{20}-(3^{1/2}/3)\varphi_{23}+(1/3)\varphi_{24}\right]$$

$$H_{r,\rho}\varphi_{14}=(\kappa_r/\kappa)\left[-(3^{1/2}/18)\varphi_4-(2^{1/2}/2)\varphi_7-(6^{1/2}/18)\varphi_8\right.$$
$$\left.-(6^{1/2}/12)\varphi_{15}-(6^{1/2}/36)\varphi_{19}-(1/2)\varphi_{21}-(3^{1/2}/18)\varphi_{22}\right]$$
$$+(\kappa_\rho/\kappa)\left[-(3^{1/2}/9)\varphi_3+(2^{1/2}/3)\varphi_5-(6^{1/2}/9)\varphi_6+(6^{1/2}/6)\varphi_{16}\right.$$
$$\left.-(6^{1/2}/18)\varphi_{20}+(1/3)\varphi_{23}-(3^{1/2}/9)\varphi_{24}\right]$$

$$H_{r,\rho}\varphi_{15}=(\kappa_r/\kappa)\left[(3^{1/2}/6)\varphi_9+(15^{1/2}/10)\varphi_{12}-3(2^{1/2}/4)\varphi_{13}\right.$$
$$\left.-(6^{1/2}/12)\varphi_{14}+(10^{1/2}/10)\varphi_{26}\right]$$
$$+(\kappa_\rho/\kappa)\left[2^{1/2}\varphi_{10}+(15^{1/2}/15)\varphi_{25}+(35^{1/2}/5)\varphi_{28}\right.$$
$$\left.+(10^{1/2}/10)\varphi_{29}-(30^{1/2}/30)\varphi_{30}\right]$$

$$H_{r,\rho}\varphi_{16}=(\kappa_r/\kappa)\big[(2^{1/2}/2)\varphi_{11}-(15^{1/2}/30)\varphi_{25}+(35^{1/2}/10)\varphi_{28}$$
$$+3(10^{1/2}/20)\varphi_{29}+(30^{1/2}/60)\varphi_{30}\big]$$
$$+(\kappa_\rho/\kappa)\big[-(3^{1/2}/3)\varphi_9+(15^{1/2}/5)\varphi_{12}-(2^{1/2}/2)\varphi_{13}$$
$$+(6^{1/2}/6)\varphi_{14}+(10^{1/2}/5)\varphi_{27}\big]$$

$$H_{r,\rho}\varphi_{17}=(\kappa_r/\kappa)\big[(2^{1/2}/2)\varphi_4-(3^{1/2}/6)\varphi_7-(1/2)\varphi_8$$
$$-\varphi_{19}+(6^{1/2}/6)\varphi_{21}+(2^{1/2}/2)\varphi_{22}\big]$$
$$+(\kappa_\rho/\kappa)\big[(2^{1/2}/3)\varphi_3+(3^{1/2}/3)\varphi_5-(1/3)\varphi_6$$
$$-(2/3)\varphi_{20}-(6^{1/2}/3)\varphi_{23}+(2^{1/2}/3)\varphi_{24}\big]$$

$$H_{r,\rho}\varphi_{18}=(\kappa_r/\kappa)\big[(6^{1/2}/18)\varphi_4-(1/2)\varphi_7-(3^{1/2}/18)\varphi_8$$
$$-(3^{1/2}/9)\varphi_{19}+(2^{1/2}/2)\varphi_{21}+(6^{1/2}/18)\varphi_{22}\big]$$
$$+(\kappa_\rho/\kappa)\big[-(6^{1/2}/9)\varphi_3-(1/3)\varphi_5+(3^{1/2}/9)\varphi_6$$
$$+2(3^{1/2}/9)\varphi_{20}+(2^{1/2}/3)\varphi_{23}-(6^{1/2}/9)\varphi_{24}\big]$$

$$H_{r,\rho}\varphi_{19}=(\kappa_r/\kappa)\big[-5(3^{1/2}/18)\varphi_9+(15^{1/2}/90)\varphi_{12}-(2^{1/2}/4)\varphi_{13}$$
$$-(6^{1/2}/36)\varphi_{14}-\varphi_{17}-(3^{1/2}/9)\varphi_{18}-(10^{1/2}/10)\varphi_{26}\big]$$
$$+(\kappa_\rho/\kappa)\big[-(2^{1/2}/3)\varphi_{10}-(15^{1/2}/3)\varphi_{25}+(35^{1/2}/15)\varphi_{28}$$
$$+(10^{1/2}/10)\varphi_{29}-(30^{1/2}/30)\varphi_{30}+2(15^{1/2}/15)\varphi_{33}-2(5^{1/2}/15)\varphi_{34}\big]$$

$$H_{r,\rho}\varphi_{20}=(\kappa_r/\kappa)\big[(2^{1/2}/6)\varphi_{11}-(15^{1/2}/6)\varphi_{25}-(35^{1/2}/30)\varphi_{28}$$
$$-3(10^{1/2}/20)\varphi_{29}-(30^{1/2}/60)\varphi_{30}+(15^{1/2}/5)\varphi_{33}+(5^{1/2}/15)\varphi_{34}\big]$$
$$+(\kappa_\rho/\kappa)\big[-5(3^{1/2}/9)\varphi_9-(15^{1/2}/45)\varphi_{12}+(2^{1/2}/6)\varphi_{13}$$
$$-(6^{1/2}/18)\varphi_{14}-(2/3)\varphi_{17}+2(3^{1/2}/9)\varphi_{18}+(10^{1/2}/5)\varphi_{27}\big]$$

$$H_{r,\rho}\varphi_{21}=(\kappa_r/\kappa)\big[-(2^{1/2}/4)\varphi_9-(10^{1/2}/20)\varphi_{12}-(3^{1/2}/6)\varphi_{13}$$

$$-(1/2)\varphi_{14}+(6^{1/2}/6)\varphi_{17}+(2^{1/2}/2)\varphi_{18}+3(15^{1/2}/10)\varphi_{26}\big]$$

$$+(\kappa_\rho/\kappa)\big[-(3^{1/2}/3)\varphi_{10}+(10^{1/2}/10)\varphi_{25}+(210^{1/2}/30)\varphi_{28}$$

$$-(15^{1/2}/5)\varphi_{29}+(5^{1/2}/5)\varphi_{30}+(10^{1/2}/5)\varphi_{33}-(30^{1/2}/15)\varphi_{34}\big]$$

$$H_{r,\rho}\varphi_{22}=(\kappa_r/\kappa)\big[-(6^{1/2}/36)\varphi_9-(30^{1/2}/180)\varphi_{12}-(1/2)\varphi_{13}$$

$$-(3^{1/2}/18)\varphi_{14}+(2^{1/2}/2)\varphi_{17}+(6^{1/2}/18)\varphi_{18}+(5^{1/2}/10)\varphi_{26}\big]$$

$$+(\kappa_\rho/\kappa)\big[(1/3)\varphi_{10}-(30^{1/2}/30)\varphi_{25}-(70^{1/2}/30)\varphi_{28}$$

$$+(5^{1/2}/5)\varphi_{29}-(15^{1/2}/15)\varphi_{30}-(30^{1/2}/15)\varphi_{33}+(10^{1/2}/15)\varphi_{34}\big]$$

$$H_{r,\rho}\varphi_{23}=(\kappa_r/\kappa)\big[-(3^{1/2}/2)\varphi_{11}-3(10^{1/2}/20)\varphi_{25}+(210^{1/2}/20)\varphi_{28}$$

$$-(15^{1/2}/10)\varphi_{29}-3(5^{1/2}/10)\varphi_{30}-(10^{1/2}/10)\varphi_{33}-(30^{1/2}/10)\varphi_{34}\big]$$

$$+(\kappa_\rho/\kappa)\big[(2^{1/2}/6)\varphi_9-(10^{1/2}/30)\varphi_{12}-(3^{1/2}/3)\varphi_{13}+(1/3)\varphi_{14}$$

$$-(6^{1/2}/3)\varphi_{17}+(2^{1/2}/3)\varphi_{18}+(15^{1/2}/5)\varphi_{27}\big]$$

$$H_{r,\rho}\varphi_{24}=(\kappa_r/\kappa)\big[-(1/6)\varphi_{11}-(30^{1/2}/60)\varphi_{25}+(70^{1/2}/60)\varphi_{28}$$

$$-3(5^{1/2}/10)\varphi_{29}-(15^{1/2}/30)\varphi_{30}-(30^{1/2}/10)\varphi_{33}-(10^{1/2}/30)\varphi_{34}\big]$$

$$+(\kappa_\rho/\kappa)\big[-(6^{1/2}/18)\varphi_9+(30^{1/2}/90)\varphi_{12}+(1/3)\varphi_{13}$$

$$-(3^{1/2}/9)\varphi_{14}+(2^{1/2}/3)\varphi_{17}-(6^{1/2}/9)\varphi_{18}-(5^{1/2}/5)\varphi_{27}\big]$$

APPENDIX C: GENERALIZED BASIS FUNCTIONS

$$\varphi_{16N+1}=N_0(2N+1)^{1/2}(P^\tau)^N\kappa_r^N\kappa_\rho^N j_{N,N}r^N\rho^N P_N(\cos\zeta)$$

$$\cdot\big[(+)^{\tau\,2b}(1)+(-)^{\tau\,2c}(1)\big]$$

$$\varphi_{16N+2}=N_0[(2N+1)/3]^{1/2}(P^\tau)^N\kappa_r^N\kappa_\rho^N j_{N,N}$$

$$\cdot r^N\rho^N P_N(\cos\zeta)\left[3(+)^{\tau\,2b}(1)-(-)^{\tau\,2c}(1)\right]$$

$$\varphi_{16N+3}=iN_0[4/3(2N+2)]^{1/2}(P^\tau)^N(-)^\tau\kappa_r^{N+1}\kappa_\rho^N j_{N+1,N}$$

$$\cdot\left[{}^4(\mathbf{r})r^N\rho^N P'_{N+1}(\cos\zeta)-{}^4(\boldsymbol{\rho})r^{N+1}\rho^{N-1}P'_N(\cos\zeta)\right]$$

$$\varphi_{16N+4}=iN_0[4/3(2N+2)]^{1/2}(P^\tau)^N(+)^\tau\kappa_r^N\kappa_\rho^{N+1} j_{N,N+1}$$

$$\cdot\left[{}^4(\boldsymbol{\rho})r^N\rho^N P'_{N+1}(\cos\zeta)-{}^4(\mathbf{r})r^{N-1}\rho^{N+1}P'_N(\cos\zeta)\right]$$

$$\varphi_{16N+5}=iN_0[2/(2N+2)]^{1/2}(P^\tau)^N\kappa_r^{N+1}\kappa_\rho^N j_{N+1,N}$$

$$\cdot\left\{\left[(+)^{\tau\,2b}(\mathbf{r})+(-)^{\tau\,2c}(\mathbf{r})\right]r^N\rho^N P'_{N+1}(\cos\zeta)\right.$$

$$\left.-\left[(+)^{\tau\,2b}(\boldsymbol{\rho})+(-)^{\tau\,2c}(\boldsymbol{\rho})\right]r^{N+1}\rho^{N-1}P'_N(\cos\zeta)\right\}$$

$$\varphi_{16N+6}=iN_0[2/3(2N+2)]^{1/2}(P^\tau)^N\kappa_r^{N+1}\kappa_\rho^N j_{N+1,N}$$

$$\cdot\left\{\left[3(+)^{\tau\,2b}(\mathbf{r})-(-)^{\tau\,2c}(\mathbf{r})\right]r^N\rho^N P'_{N+1}(\cos\zeta)\right.$$

$$\left.-\left[3(+)^{\tau\,2b}(\boldsymbol{\rho})-(-)^{\tau\,2c}(\boldsymbol{\rho})\right]r^{N+1}\rho^{N-1}P'_N(\cos\zeta)\right\}$$

$$\varphi_{16N+7}=iN_0[2/(2N+2)]^{1/2}(P^\tau)^N\kappa_r^N\kappa_\rho^{N+1} j_{N,N+1}$$

$$\cdot\left\{\left[(-)^{\tau\,2b}(\boldsymbol{\rho})+(+)^{\tau\,2c}(\boldsymbol{\rho})\right]r^N\rho^N P'_{N+1}(\cos\zeta)\right.$$

$$\left.-\left[(-)^{\tau\,2b}(\mathbf{r})+(+)^{\tau\,2c}(\mathbf{r})\right]r^{N-1}\rho^{N+1}P'_N(\cos\zeta)\right\}$$

$$\varphi_{16N+8}=iN_0[2/3(2N+2)]^{1/2}(P^\tau)^N\kappa_r^N\kappa_\rho^{N+1} j_{N,N+1}$$

$$\cdot\left\{\left[3(-)^{\tau\,2b}(\boldsymbol{\rho})-(+)^{\tau\,2c}(\boldsymbol{\rho})\right]r^N\rho^N P'_{N+1}(\cos\zeta)\right.$$

$$\left.-\left[3(-)^{\tau\,2b}(\mathbf{r})-(+)^{\tau\,2c}(\mathbf{r})\right]r^{N-1}\rho^{N+1}P'_N(\cos\zeta)\right\}$$

$$\varphi_{16N+9}=N_0[8(2N+3)/3(2N+2)(2N+4)]^{1/2}(P^\tau)^N(+)^\tau$$

$$\cdot\kappa_r^{N+1}\kappa_\rho^{N+1}j_{N+1,N+1}\,{}^4(i\mathbf{r}\times\boldsymbol{\rho})r^N\rho^N P'_{N+1}(\cos\zeta)$$

$$\varphi_{16N+10}=N_0[16/(2N+2)(2N+3)(2N+4)]^{1/2}(P^\tau)^N(-)^\tau\kappa_r^{N+2}\kappa_\rho^N$$

$$\cdot j_{N+2,N}[\,{}^4(\mathbf{rr})r^N\rho^N P''_{N+2}(\cos\zeta)-{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})r^{N+1}\rho^{N-1}P''_{N+1}(\cos\zeta)$$

$$+\,{}^4(\boldsymbol{\rho\rho})r^{N+2}\rho^{N-2}P''_N(\cos\zeta)]$$

$$\varphi_{16N+11}=N_0[16/(2N+2)(2N+3)(2N+4)]^{1/2}(P^\tau)^N(-)^\tau\kappa_r^N\kappa_\rho^{N+2}$$

$$\cdot j_{N,N+2}[\,{}^4(\boldsymbol{\rho\rho})r^N\rho^N P''_{N+2}(\cos\zeta)-{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})r^{N-1}\rho^{N+1}P''_{N+1}(\cos\zeta)$$

$$+\,{}^4(\mathbf{rr})r^{N-2}\rho^{N+2}P''_N(\cos\zeta)]$$

$$\varphi_{16N+12}=N_0[24/(2N+1)(2N+2)(2N+3)(2N+4)(2N+5)]^{1/2}$$

$$\cdot(P^\tau)^N(+)^\tau\kappa_r^{N+1}\kappa_\rho^{N+1}j_{N+1,N+1}$$

$$\cdot\{\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})r^N\rho^N[(2N+1)P''_{N+2}(\cos\zeta)+(2N+5)P''_N(\cos\zeta)]$$

$$-2(2N+3)[\,{}^4(\mathbf{rr})r^{N-1}\rho^{N+1}+{}^4(\boldsymbol{\rho\rho})r^{N+1}\rho^{N-1}]P''_{N+1}(\cos\zeta)\}$$

$$\varphi_{16N+13}=N_0[4(2N+3)/(2N+2)(2N+4)]^{1/2}(P^\tau)^N\kappa_r^{N+1}\kappa_\rho^{N+1}$$

$$\cdot j_{N+1,N+1}[(-1)^{\tau}\,{}^{2b}(i\mathbf{r}\times\boldsymbol{\rho})+(+)^{\tau}\,{}^{2c}(i\mathbf{r}\times\boldsymbol{\rho})]\cdot r^N\rho^N P'_{N+1}(\cos\zeta)$$

$$\varphi_{16N+14}=N_0[4(2N+3)/3(2N+2)(2N+4)]^{1/2}(P^\tau)^N\kappa_r^{N+1}\kappa_\rho^{N+1}$$

$$\cdot j_{N+1,N+1}[3(-)^{\tau}\,{}^{2b}(i\mathbf{r}\times\boldsymbol{\rho})-(+)^{\tau}\,{}^{2c}(i\mathbf{r}\times\boldsymbol{\rho})]\cdot r^N\rho^N P'_{N+1}(\cos\zeta)$$

$$\varphi_{16N+15}=iN_0[64/(2N+2)(2N+4)(2N+6)]^{1/2}(P^\tau)^N(+)^\tau$$

$$\cdot\kappa_r^{N+2}\kappa_\rho^{N+1}j_{N+2,N+1}[\,{}^4(i\mathbf{rr}\times\boldsymbol{\rho})r^N\rho^N P''_{N+2}(\cos\zeta)$$

$$-\,{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})r^{N+1}\rho^{N-1}P''_{N+1}(\cos\zeta)]$$

$$\varphi_{16N+16} = iN_0[64/(2N+2)(2N+4)(2N+6)]^{1/2}(P^\tau)^N(-)^\tau \cdot \kappa_r^{N+1}\kappa_\rho^{N+2} j_{N+1,N+2}[{}^4(i\rho\mathbf{r}\times\boldsymbol{\rho})r^N\rho^N P''_{N+2}(\cos\zeta) - {}^4(i\mathbf{rr}\times\boldsymbol{\rho})r^{N-1}\rho^{N+1}P''_{N+1}(\cos\zeta)]$$

APPENDIX D: GENERALIZED $H_{r,\rho}$ MATRIX

$$H_{r,\rho}\varphi_{16N+1} = [(2N)/(2N+1)]^{1/2} \times \{(\kappa_r/\kappa)[(3/4)^{1/2}\varphi_{16N-12} - (1/8)^{1/2}\varphi_{16N-9} - (3/8)^{1/2}\varphi_{16N-8}] + (\kappa_\rho/\kappa)[(1/3)^{1/2}\varphi_{16N-13} + (1/2)^{1/2}\varphi_{16N-11} - (1/6)^{1/2}\varphi_{16N-10}]\} + [(2N+2)/(2N+1)]^{1/2}\{(\kappa_r/\kappa)[-(3/4)^{1/2}\varphi_{16N+3} + (1/8)^{1/2}\varphi_{16N+5} + (3/8)^{1/2}\varphi_{16N+6}] + (\kappa_\rho/\kappa)[-(1/3)^{1/2}\varphi_{16N+4} - (1/2)^{1/2}\varphi_{16N+7} + (1/6)^{1/2}\varphi_{16N+8}]\}$$

$$H_{r,\rho}\varphi_{16N+2} = [(2N)/(2N+1)]^{1/2} \times \{(\kappa_r/\kappa)[(1/6)\varphi_{16N-12} - (3/8)^{1/2}\varphi_{16N-9} - (1/72)^{1/2}\varphi_{16N-8}] + (\kappa_\rho/\kappa)[-(1/3)\varphi_{16N-13} - (1/6)^{1/2}\varphi_{16N-11} + (1/18)^{1/2}\varphi_{16N-10}]\} + [(2N+2)/(2N+1)]^{1/2}\{(\kappa_r/\kappa)[-(1/6)\varphi_{16N+3} + (3/8)^{1/2}\varphi_{16N+5} + (1/72)^{1/2}\varphi_{16N+6}] + (\kappa_\rho/\kappa)[(1/3)\varphi_{16N+4} + (1/6)^{1/2}\varphi_{16N+7} - (1/18)^{1/2}\varphi_{16N+8}]\}$$

$$H_{r,\rho}\varphi_{16N+3} = [(2N)/(2N+1)]^{1/2}\{(\kappa_r/\kappa)[-(25/72)^{1/2}\varphi_{16N-7} - (3/16)^{1/2}\varphi_{16N-3} - (1/12)\varphi_{16N-2}] + (\kappa_\rho/\kappa)[-(1/3)^{1/2}\varphi_{16N-6}]\} + [(2N-1)(2N)/(2N+1)(2N+3)]^{1/2}(\kappa_r/\kappa)(1/72)^{1/2}\varphi_{16N-4} + [(2N+2)/(2N+1)]^{1/2}(\kappa_r/\kappa)[-(3/4)^{1/2}\varphi_{16N+1} - (1/6)\varphi_{16N+2}]$$

$$+\left[(2N+4)/(2N+3)\right]^{1/2}\left\{(\kappa_r/\kappa)\left[-(1/12)^{1/2}\varphi_{16N+10}\right]\right.$$

$$\left.+(\kappa_\rho/\kappa)\left[-(25/18)^{1/2}\varphi_{16N+9}+(1/12)^{1/2}\varphi_{16N+13}-(1/6)\varphi_{16N+14}\right]\right\}$$

$$+\left[(2N+4)(2N+5)/(2N+1)(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)(1/18)^{1/2}\varphi_{16N+12}$$

$$+\left[(2N+2)/(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)\left[(1/3)^{1/2}\varphi_{16N+17}-(1/3)\varphi_{16N+18}\right]$$

$$H_{r,\rho}\varphi_{16N+4}=\left[(2N)/(2N+1)\right]^{1/2}\left\{(\kappa_r/\kappa)(1/12)^{1/2}\varphi_{16N-5}\right.$$

$$\left.+(\kappa_\rho/\kappa)\left[-(25/18)^{1/2}\varphi_{16N-7}+(1/12)^{1/2}\varphi_{16N-3}-(1/6)\varphi_{16N-2}\right]\right\}$$

$$+\left[(2N-1)(2N)/(2N+1)(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)\left[-(1/18)^{1/2}\varphi_{16N-4}\right]$$

$$+\left[(2N+2)/(2N+1)\right]^{1/2}(\kappa_\rho/\kappa)\left[-(1/3)^{1/2}\varphi_{16N+1}+(1/3)\varphi_{16N+2}\right]$$

$$+\left[(2N+4)/(2N+3)\right]^{1/2}\left\{(\kappa_r/\kappa)\left[-(25/72)^{1/2}\varphi_{16N+9}\right.\right.$$

$$\left.\left.-(3/16)^{1/2}\varphi_{16N+13}-(1/12)\varphi_{16N+14}\right]+(\kappa_\rho/\kappa)(1/3)^{1/2}\varphi_{16N+11}\right\}$$

$$+\left[(2N+4)(2N+5)/(2N+1)(2N+3)\right]^{1/2}(\kappa_r/\kappa)\left[-(1/72)^{1/2}\varphi_{16N+12}\right]$$

$$+\left[(2N+2)/(2N+3)\right]^{1/2}(\kappa_r/\kappa)\left[(3/4)^{1/2}\varphi_{16N+17}+(1/6)\varphi_{16N+18}\right]$$

$$H_{r,\rho}\varphi_{16N+5}=\left[(2N)/(2N+1)\right]^{1/2}\left\{(\kappa_r/\kappa)\left[-(3/16)^{1/2}\varphi_{16N-7}\right.\right.$$

$$\left.\left.-(1/8)^{1/2}\varphi_{16N-3}-(3/8)^{1/2}\varphi_{16N-2}\right]+(\kappa_\rho/\kappa)\left[-(1/2)^{1/2}\varphi_{16N-6}\right]\right\}$$

$$+\left[(2N-1)(2N)/(2N+1)(2N+3)\right]^{1/2}(\kappa_r/\kappa)\left[-(3/16)^{1/2}\varphi_{16N-4}\right]$$

$$+\left[(2N+2)/(2N+1)\right]^{1/2}(\kappa_r/\kappa)\left[(1/8)^{1/2}\varphi_{16N+1}+(3/8)^{1/2}\varphi_{16N+2}\right]$$

$$+\left[(2N+4)/(2N+3)\right]^{1/2}\left\{(\kappa_r/\kappa)(9/8)^{1/2}\varphi_{16N+10}\right.$$

$$\left.+(\kappa_\rho/\kappa)\left[(1/12)^{1/2}\varphi_{16N+9}-(1/2)^{1/2}\varphi_{16N+13}+(1/6)^{1/2}\varphi_{16N+14}\right]\right\}$$

$$+\left[(2N+4)(2N+5)/(2N+1)(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)(1/12)^{1/2}\varphi_{16N+12}$$

$$+\left[(2N+2)/(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)\left[(1/2)^{1/2}\varphi_{16N+17}-(1/6)^{1/2}\varphi_{16N+18}\right]$$

$$H_{r,\rho}\varphi_{16N+6}=\left[(2N)/(2N+1)\right]^{1/2}\Big\{(\kappa_r/\kappa)\Big[-(1/12)\varphi_{16N-7}$$

$$-(3/8)^{1/2}\varphi_{16N-3}-(1/72)^{1/2}\varphi_{16N-2}\Big]+(\kappa_\rho/\kappa)(1/6)^{1/2}\varphi_{16N-6}\Big\}$$

$$+\left[(2N-1)(2N)/(2N+1)(2N+3)\right]^{1/2}(\kappa_r/\kappa)\left[-(1/12)\varphi_{16N-4}\right]$$

$$+\left[(2N+2)/(2N+1)\right]^{1/2}(\kappa_r/\kappa)\left[(3/8)^{1/2}\varphi_{16N+1}+(1/72)^{1/2}\varphi_{16N+2}\right]$$

$$+\left[(2N+4)/(2N+3)\right]^{1/2}\Big\{(\kappa_r/\kappa)(1/24)^{1/2}\varphi_{16N+10}$$

$$+(\kappa_\rho/\kappa)\Big[-(1/6)\varphi_{16N+9}+(1/6)^{1/2}\varphi_{16N+13}-(1/18)^{1/2}\varphi_{16N+14}\Big]\Big\}$$

$$+\left[(2N+4)(2N+5)/(2N+1)(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)\left[-(1/6)\varphi_{16N+12}\right]$$

$$+\left[(2N+2)/(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)\left[-(1/6)^{1/2}\varphi_{16N+17}+(1/18)^{1/2}\varphi_{16N+18}\right]$$

$$H_{r,\rho}\varphi_{16N+7}=\left[(2N)/(2N+1)\right]^{1/2}\Big\{(\kappa_r/\kappa)\Big[-(9/8)^{1/2}\varphi_{16N-5}\Big]$$

$$+(\kappa_\rho/\kappa)\Big[(1/12)^{1/2}\varphi_{16N-7}-(1/2)^{1/2}\varphi_{16N-3}+(1/6)^{1/2}\varphi_{16N-2}\Big]\Big\}$$

$$+\left[(2N-1)(2N)/(2N+1)(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)\left[-(1/12)^{1/2}\varphi_{16N-4}\right]$$

$$+\left[(2N+2)/(2N+1)\right]^{1/2}(\kappa_\rho/\kappa)\left[-(1/2)^{1/2}\varphi_{16N+1}+(1/6)^{1/2}\varphi_{16N+2}\right]$$

$$+\left[(2N+4)/(2N+3)\right]^{1/2}\Big\{(\kappa_r/\kappa)\Big[-(3/16)^{1/2}\varphi_{16N+9}$$

$$-(1/8)^{1/2}\varphi_{16N+13}-(3/8)^{1/2}\varphi_{16N+14}\Big]+(\kappa_\rho/\kappa)(1/2)^{1/2}\varphi_{16N+11}\Big\}$$

$$+\left[(2N+4)(2N+5)/(2N+1)(2N+3)\right]^{1/2}(\kappa_r/\kappa)(3/16)^{1/2}\varphi_{16N+12}$$

$$+\left[(2N+2)/(2N+3)\right]^{1/2}(\kappa_r/\kappa)\left[-(1/8)^{1/2}\varphi_{16N+17}-(3/8)^{1/2}\varphi_{16N+18}\right]$$

$$H_{r,\rho}\varphi_{16N+8}=\left[(2N)/(2N+1)\right]^{1/2}\Big\{(\kappa_r/\kappa)\Big[-(1/24)^{1/2}\varphi_{16N-5}\Big]$$

$$+(\kappa_\rho/\kappa)\Big[-(1/6)\varphi_{16N-7}+(1/6)^{1/2}\varphi_{16N-3}-(1/18)^{1/2}\varphi_{16N-2}\Big]\Big\}$$

$$+\left[(2N-1)(2N)/(2N+1)(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)(1/6)\varphi_{16N-4}$$

$$+\left[(2N+2)/(2N+1)\right]^{1/2}(\kappa_\rho/\kappa)\left[(1/6)^{1/2}\varphi_{16N+1}-(1/18)^{1/2}\varphi_{16N+2}\right]$$

$$+\left[(2N+4)/(2N+3)\right]^{1/2}\left\{(\kappa_r/\kappa)\left[-(1/12)\varphi_{16N+9}\right.\right.$$

$$\left.\left.-(3/8)^{1/2}\varphi_{16N+13}-(1/72)^{1/2}\varphi_{16N+14}\right]+(\kappa_\rho/\kappa)\left[-(1/6)^{1/2}\varphi_{16N+11}\right]\right\}$$

$$+\left[(2N+4)(2N+5)/(2N+1)(2N+3)\right]^{1/2}(\kappa_r/\kappa)(1/12)\varphi_{16N+12}$$

$$+\left[(2N+2)/(2N+3)\right]^{1/2}(\kappa_r/\kappa)\left[-(3/8)^{1/2}\varphi_{16N+17}-(1/72)^{1/2}\varphi_{16N+18}\right]$$

$$H_{r,\rho}\varphi_{16N+9}=\left[(2N)/(2N+3)\right]^{1/2}$$

$$\times\left\{(\kappa_r/\kappa)\left[-(1/24)^{1/2}\varphi_{16N}\right]+(\kappa_\rho/\kappa)(1/6)^{1/2}\varphi_{16N-1}\right\}$$

$$+\left[(2N+4)/(2N+3)\right]^{1/2}\left\{(\kappa_r/\kappa)\left[-(25/72)^{1/2}\varphi_{16N+4}\right.\right.$$

$$\left.-(3/16)^{1/2}\varphi_{16N+7}-(1/12)\varphi_{16N+8}\right]$$

$$\left.+(\kappa_\rho/\kappa)\left[-(25/18)^{1/2}\varphi_{16N+3}+(1/12)^{1/2}\varphi_{16N+5}-(1/6)\varphi_{16N+6}\right]\right\}$$

$$+\left[(2N+6)/(2N+3)\right]^{1/2}\left\{(\kappa_r/\kappa)(1/24)^{1/2}\varphi_{16N+15}\right.$$

$$\left.+(\kappa_\rho/\kappa)\left[-(1/6)^{1/2}\varphi_{16N+16}\right]\right\}$$

$$+\left[(2N+2)/(2N+3)\right]^{1/2}\left\{(\kappa_r/\kappa)\left[-(25/72)^{1/2}\varphi_{16N+19}\right.\right.$$

$$\left.-(3/16)^{1/2}\varphi_{16N+21}-(1/12)\varphi_{16N+22}\right]$$

$$\left.+(\kappa_\rho/\kappa)\left[-(25/18)^{1/2}\varphi_{16N+20}+(1/12)^{1/2}\varphi_{16N+23}-(1/6)\varphi_{16N+24}\right]\right\}$$

$$H_{r,\rho}\varphi_{16N+10}=\left[(2N)/(2N+3)\right]^{1/2}(\kappa_r/\kappa)(1/2)\varphi_{16N-1}$$

$$+\left[(2N+4)/(2N+3)\right]^{1/2}(\kappa_r/\kappa)\left[-(1/12)^{1/2}\varphi_{16N+3}\right.$$

$$\left.+(9/8)^{1/2}\varphi_{16N+5}+(1/24)^{1/2}\varphi_{16N+6}\right]$$

$$+\left[(2N+6)/(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)\varphi_{16N+15}+\left[(2N+2)/(2N+3)\right]^{1/2}$$

$$\times(\kappa_\rho/\kappa)\left[-(1/3)^{1/2}\varphi_{16N+19}-(1/2)^{1/2}\varphi_{16N+21}+(1/6)^{1/2}\varphi_{16N+22}\right]$$

$$H_{r,\rho}\varphi_{16N+11} = \left[(2N)/(2N+3)\right]^{1/2}(\kappa_\rho/\kappa)\varphi_{16N} + \left[(2N+4)/(2N+3)\right]^{1/2}$$

$$\times(\kappa_\rho/\kappa)\left[(1/3)^{1/2}\varphi_{16N+4} + (1/2)^{1/2}\varphi_{16N+7} - (1/6)^{1/2}\varphi_{16N+8}\right]$$

$$+\left[(2N+6)/(2N+3)\right]^{1/2}(\kappa_r/\kappa)(1/2)\varphi_{16N+16} + \left[(2N+2)/(2N+3)\right]^{1/2}$$

$$\times(\kappa_r/\kappa)\left[(1/12)^{1/2}\varphi_{16N+20} - (9/8)^{1/2}\varphi_{16N+23} - (1/24)^{1/2}\varphi_{16N+24}\right]$$

$$H_{r,\rho}\varphi_{16N+12} = \left[(2N)(2N+5)/(2N+1)(2N+3)\right]^{1/2}$$

$$\cdot\left\{(\kappa_r/\kappa)(3/8)^{1/2}\varphi_{16N} + (\kappa_\rho/\kappa)(3/2)^{1/2}\varphi_{16N-1}\right\}$$

$$+\left[(2N+4)(2N+5)/(2N+1)(2N+3)\right]^{1/2}$$

$$\times\left\{(\kappa_r/\kappa)\cdot\left[-(1/72)^{1/2}\varphi_{16N+4} + (3/16)^{1/2}\varphi_{16N+7} + (1/12)\varphi_{16N+8}\right]\right.$$

$$\left.+(\kappa_\rho/\kappa)\left[(1/18)^{1/2}\varphi_{16N+3} + (1/12)^{1/2}\varphi_{16N+5} - (1/6)\varphi_{16N+6}\right]\right\}$$

$$+\left[(2N+1)(2N+6)/(2N+3)(2N+5)\right]^{1/2}\cdot\left\{(\kappa_r/\kappa)(3/8)^{1/2}\varphi_{16N+15}\right.$$

$$\left.+(\kappa_\rho/\kappa)(3/2)^{1/2}\varphi_{16N+16}\right\} + \left[(2N+1)(2N+2)/(2N+3)(2N+5)\right]^{1/2}$$

$$\times\left\{(\kappa_r/\kappa)\cdot\left[(1/72)^{1/2}\varphi_{16N+19} - (3/16)^{1/2}\varphi_{16N+21} - (1/12)\varphi_{16N+22}\right]\right.$$

$$\left.+(\kappa_\rho/\kappa)\left[-(1/18)^{1/2}\varphi_{16N+20} - (1/12)^{1/2}\varphi_{16N+23} + (1/6)\varphi_{16N+24}\right]\right\}$$

$$H_{r,\rho}\varphi_{16N+13} = \left[(2N)/(2N+3)\right]^{1/2}$$

$$\times\left\{(\kappa_r/\kappa)(3/4)\varphi_{16N} + (\kappa_\rho/\kappa)(1/2)\varphi_{16N-1}\right\}$$

$$+\left[(2N+4)/(2N+3)\right]^{1/2}\left\{(\kappa_r/\kappa)\left[-(3/16)^{1/2}\varphi_{16N+4}\right.\right.$$

$$\left.-(1/8)^{1/2}\varphi_{16N+7} - (3/8)^{1/2}\varphi_{16N+8}\right]$$

$$\left.+(\kappa_\rho/\kappa)\left[(1/12)^{1/2}\varphi_{16N+3} - (1/2)^{1/2}\varphi_{16N+5} + (1/6)^{1/2}\varphi_{16N+6}\right]\right\}$$

$$+\left[(2N+6)/(2N+3)\right]^{1/2}\left\{(\kappa_r/\kappa)\left[-(3/4)\varphi_{16N+15}\right]\right.$$

$$\left.+(\kappa_\rho/\kappa)\left[-(1/2)\varphi_{16N+16}\right]\right\}$$

$$+[(2N+2)/(2N+3)]^{1/2}\{(\kappa_r/\kappa)[-(3/16)^{1/2}\varphi_{16N+19}$$

$$-(1/8)^{1/2}\varphi_{16N+21}-(3/8)^{1/2}\varphi_{16N+22}]$$

$$+(\kappa_\rho/\kappa)[(1/12)^{1/2}\varphi_{16N+20}-(1/2)^{1/2}\varphi_{16N+23}+(1/6)^{1/2}\varphi_{16N+24}]\}$$

$$H_{r,\rho}\varphi_{16N+14}=[(2N)/(2N+3)]^{1/2}$$

$$\times\{(\kappa_r/\kappa)(1/48)^{1/2}\varphi_{16N}+(\kappa_\rho/\kappa)[-(1/12)^{1/2}\varphi_{16N-1}]\}$$

$$+[(2N+4)/(2N+3)]^{1/2}\{(\kappa_r/\kappa)[-(1/12)\varphi_{16N+4}$$

$$-(3/8)^{1/2}\varphi_{16N+7}-(1/72)^{1/2}\varphi_{16N+8}]$$

$$+(\kappa_\rho/\kappa)[-(1/6)\varphi_{16N+3}+(1/6)^{1/2}\varphi_{16N+5}-(1/18)^{1/2}\varphi_{16N+6}]\}$$

$$+[(2N+6)/(2N+3)]^{1/2}\{(\kappa_r/\kappa)[-(1/48)^{1/2}\varphi_{16N+15}]$$

$$+(\kappa_\rho/\kappa)(1/12)^{1/2}\varphi_{16N+6}\}+[(2N+2)/(2N+3)]^{1/2}$$

$$\times\{(\kappa_r/\kappa)[-(1/12)\varphi_{16N+19}-(3/8)^{1/2}\varphi_{16N+21}-(1/72)^{1/2}\varphi_{16N+22}]$$

$$+(\kappa_\rho/\kappa)[-(1/6)\varphi_{16N+20}+(1/6)^{1/2}\varphi_{16N+23}-(1/18)^{1/2}\varphi_{16N+24}]\}$$

$$H_{r,\rho}\varphi_{16N+15}=[(2N+6)/(2N+3)]^{1/2}\{(\kappa_r/\kappa)[(1/24)^{1/2}\varphi_{16N+9}$$

$$-(3/4)\varphi_{16N+13}-(1/48)^{1/2}\varphi_{16N+14}]+(\kappa_\rho/\kappa)\varphi_{16N+10}\}$$

$$+[(2N+1)(2N+6)/(2N+3)(2N+5)]^{1/2}(\kappa_r/\kappa)(3/8)^{1/2}\varphi_{16N+12}$$

$$+[(2N+2)/(2N+5)]^{1/2}\{(\kappa_r/\kappa)(1/2)\varphi_{16N+26}$$

$$+(\kappa_\rho/\kappa)[(1/6)^{1/2}\varphi_{16N+25}+(1/2)\varphi_{16N+29}-(1/12)^{1/2}\varphi_{16N+30}]\}$$

$$+[(2N+2)(2N+7)/(2N+3)(2N+5)]^{1/2}(\kappa_\rho/\kappa)(3/2)^{1/2}\varphi_{16N+28}$$

$$H_{r,\rho}\varphi_{16N+16}=[(2N+6)/(2N+3)]^{1/2}\{(\kappa_r/\kappa)(1/2)\varphi_{16N+11}$$

$$+(\kappa_\rho/\kappa)[-(1/6)^{1/2}\varphi_{16N+9}-(1/2)\varphi_{16N+13}+(1/12)^{1/2}\varphi_{16N+14}]\}$$

$$+[(2N+1)(2N+6)/(2N+3)(2N+5)]^{1/2}(\kappa_\rho/\kappa)(3/2)^{1/2}\varphi_{16N+12}$$

$$+[(2N+2)/(2N+5)]^{1/2}\{(\kappa_r/\kappa)[-(1/24)^{1/2}\varphi_{16N+25}$$

$$+(3/4)\varphi_{16N+29}+(1/48)^{1/2}\varphi_{16N+30}]+(\kappa_\rho/\kappa)\varphi_{16N+27}\}$$

$$+[(2N+2)(2N+7)/(2N+3)(2N+5)]^{1/2}(\kappa_r/\kappa)(3/8)^{1/2}\varphi_{16N+28}$$

APPENDIX E: 32 REDUCTION FORMULAS

$$C_{17}=-3^{1/2}\nu C_1$$

$$C_{18}=-3^{1/2}\nu C_2$$

$$C_{19}=(1/2)^{1/2}[-3\nu C_3+C_4]$$

$$C_{20}=(1/2)^{1/2}[C_3-3\nu C_4]$$

$$C_{21}=(1/2)^{1/2}[-3\nu C_5+C_7]$$

$$C_{22}=(1/2)^{1/2}[-3\nu C_6+C_8]$$

$$C_{23}=(1/2)^{1/2}[C_5-3\nu C_7]$$

$$C_{24}=(1/2)^{1/2}[C_6-3\nu C_8]$$

$$C_{25}=-5^{1/2}\nu C_9$$

$$C_{26}=5^{1/2}[-\nu C_{10}+(2/15)^{1/2}C_{12}]$$

$$C_{27}=5^{1/2}[-\nu C_{11}+(2/15)^{1/2}C_{12}]$$

$$C_{28}=(1/21)^{1/2}[30^{1/2}(C_{10}+C_{11})-15\nu C_{12}]$$

$$C_{29}=-5^{1/2}\nu C_{13}$$

$$C_{30}=-5^{1/2}\nu C_{14}$$

$$C_{31}=(1/2)[-5\nu C_{15}+C_{16}]$$

$$C_{32}=(1/2)[C_{15}-5\nu C_{16}]$$

$$C_{33}=(5/4)^{1/2}(3\nu^2-1)C_1$$

$$C_{34}=(5/4)^{1/2}(3\nu^2-1)C_2$$

$$C_{35}=(3/4)^{1/2}[(5\nu^2-1)C_3-2\nu C_4]$$

$$C_{36}=(3/4)^{1/2}[-2\nu C_3+(5\nu^2-1)C_4]$$

$$C_{37}=(3/4)^{1/2}[(5\nu^2-1)C_5-2\nu C_7]$$

$$C_{38}=(3/4)^{1/2}[(5\nu^2-1)C_6-2\nu C_8]$$

$$C_{39}=(3/4)^{1/2}[-2\nu C_5+(5\nu^2-1)C_7]$$

$$C_{40}=(3/4)^{1/2}[-2\nu C_6+(5\nu^2-1)C_8]$$

$$C_{41}=(7/8)^{1/2}(5\nu^2-1)C_9$$

$$C_{42}=(1/14)^{1/2}[(5/2)(7\nu^2-1)C_{10}+C_{11}-5(10/3)^{1/2}\nu C_{12}]$$

$$C_{43}=(1/14)^{1/2}[C_{10}+(5/2)(7\nu^2-1)C_{11}-5(10/3)^{1/2}\nu C_{12}]$$

$$C_{44}=(7/2)^{1/2}[-(10/3)^{1/2}\nu(C_{10}+C_{11})+(1/6)(25\nu^2-1)C_{12}]$$

$$C_{45}=(7/8)^{1/2}(5\nu^2-1)C_{13}$$

$$C_{46}=(7/8)^{1/2}(5\nu^2-1)C_{14}$$

$$C_{47}=(5/8)^{1/2}[(7\nu^2-1)C_{15}-2\nu C_{16}]$$

$$C_{48}=(5/8)^{1/2}[-2\nu C_{15}+(7\nu^2-1)C_{16}]$$

APPENDIX F: GENERALIZED REDUCTION FORMULAS

$$C_{16N+1}=(2N+1)^{1/2}P_N(-\nu)C_1$$

$$C_{16N+2}=(2N+1)^{1/2}P_N(-\nu)C_2$$

$$C_{16N+3}=[1/(N+1)]^{1/2}[P'_{N+1}(-\nu)C_3+P'_N(-\nu)C_4]$$

$$C_{16N+4}=[1/(N+1)]^{1/2}[P'_N(-\nu)C_3+P'_{N+1}(-\nu)C_4]$$

$$C_{16N+5}=[1/(N+1)]^{1/2}[P'_{N+1}(-\nu)C_5+P'_N(-\nu)C_7]$$

$$C_{16N+6}=[1/(N+1)]^{1/2}[P'_{N+1}(-\nu)C_6+P'_N(-\nu)C_8]$$

$$C_{16N+7}=[1/(N+1)]^{1/2}[P'_N(-\nu)C_5+P'_{N+1}(-\nu)C_7]$$

$$C_{16N+8}=[1/(N+1)]^{1/2}[P'_N(-\nu)C_6+P'_{N+1}(-\nu)C_8]$$

$$C_{16N+9}=[2(2N+3)/3(N+1)(N+2)]^{1/2}P'_{N+1}(-\nu)C_9$$

$$C_{16N+10}=[2/3(N+1)(N+2)(2N+3)]^{1/2}\Big[P''_{N+2}(-\nu)C_{10} + P''_N(-\nu)C_{11}+(10/3)^{1/2}P''_{N+1}(-\nu)C_{12}\Big]$$

$$C_{16N+11}=[2/3(N+1)(N+2)(2N+3)]^{1/2}\Big[P''_N(-\nu)C_{10} + P''_{N+2}(-\nu)C_{11}+(10/3)^{1/2}P''_{N+1}(-\nu)C_{12}\Big]$$

$$C_{16N+12}=[10/3(N+1)(N+2)(2N+1)(2N+3)(2N+5)]^{1/2} \cdot\big\{2(3/10)^{1/2}(2N+3)P''_{N+1}(-\nu)(C_{10}+C_{11}) + [(2N+1)P''_{N+2}(-\nu)+(2N+5)P''_N(-\nu)]C_{12}\big\}$$

$$C_{16N+13}=[2(2N+3)/3(N+1)(N+2)]^{1/2}P'_{N+1}(-\nu)C_{13}$$

$$C_{16N+14}=[2(2N+3)/3(N+1)(N+2)]^{1/2}P'_{N+1}(-\nu)C_{14}$$

$$C_{16N+15}=[2/3(N+1)(N+2)(N+3)]^{1/2}[P''_{N+2}(-\nu)C_{15}+P''_{N+1}(-\nu)C_{16}]$$

$$C_{16N+16}=[2/3(N+1)(N+2)(N+3)]^{1/2}[P''_{N+1}(-\nu)C_{15}+P''_{N+2}(-\nu)C_{16}]$$

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